Inverse proportion graphs (the graph of y=k/x)

Fig. 9 shows the curves $y = \frac{1}{x+2}$ and $y = x^2 + 7x + 7$.

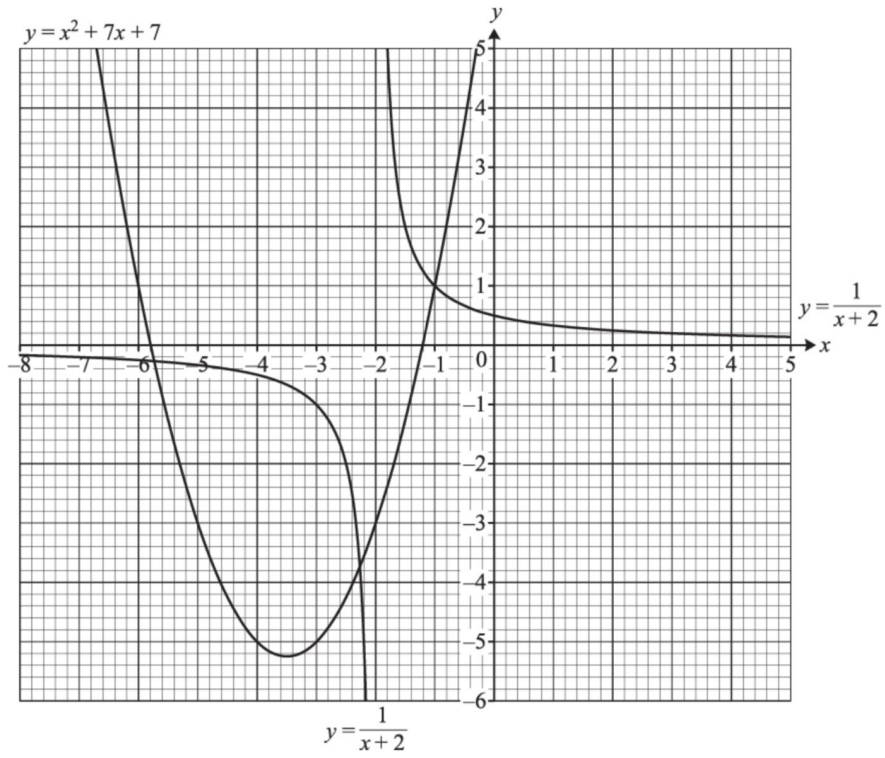


Fig.9

(i) Use Fig. 9 to estimate graphically the roots of the equation $\frac{1}{x+2} = x^2 + 7x + 7$.

[2]

(ii) Show that the equation in part **(i)** may be simplified to $x^3 + 9x^2 + 21x + 13 = 0$. Find algebraically the exact roots of this equation.

[7]

(iii) The curve $y = x^2 + 7x + 7$ is translated by $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

(*A*) Show graphically that the translated curve intersects the curve $y = \frac{1}{x+2}$ at only one point. Estimate the coordinates of this point.

[2]

(*B*) Find the equation of the translated curve, simplifying your answer.

[2]

The equation of a curve is $y = \frac{a}{(x+b)^2}$. Fig. 5 shows the curve for particular values of the constants a and b.

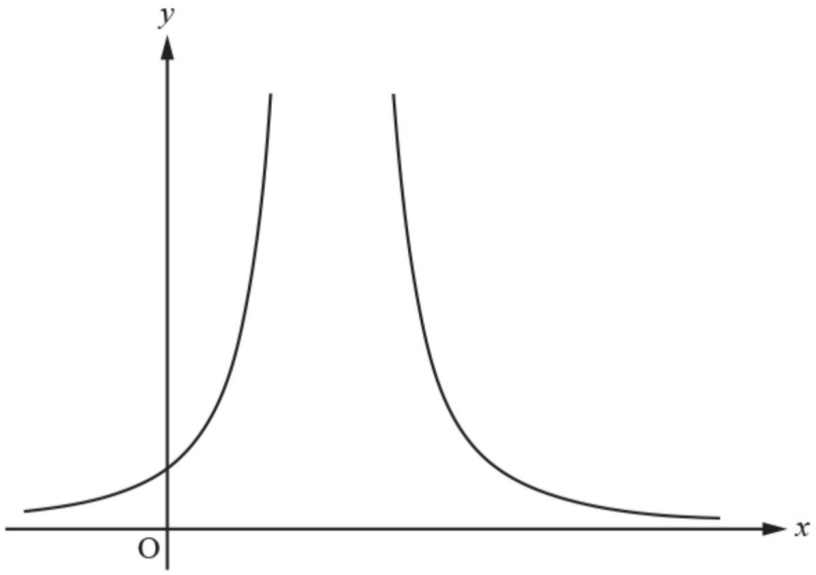


Fig. 5

(a) Write down the equations of the asymptotes, in terms of a and/or b where necessary. [2]

(b) Joe says "For all values of a and b, the curve lies above the x-axis." Determine whether Joe's statement is true or false. [2]

(c) The curve goes through the points (1, 3) and (3, 3). Find the values of a and b. [4]

Sketch and label each pair of graphs on the same set of axes showing the coordinates of any points where the graphs intersect. Write down the equations of any asymptotes.

a $y = x^2$ and $y = x^3$

b $y = x^2$ and $y = x^4$

c $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$

d $y = x$ and $y = \sqrt{x}$

e $y = x^2$ and $y = 3x^2$

f $y = \frac{1}{x}$ and $y = \frac{2}{x}$

a Describe two different transformations, each of which would map the graph of $y = \frac{1}{x}$ onto the graph of $y = \frac{1}{3x}$.

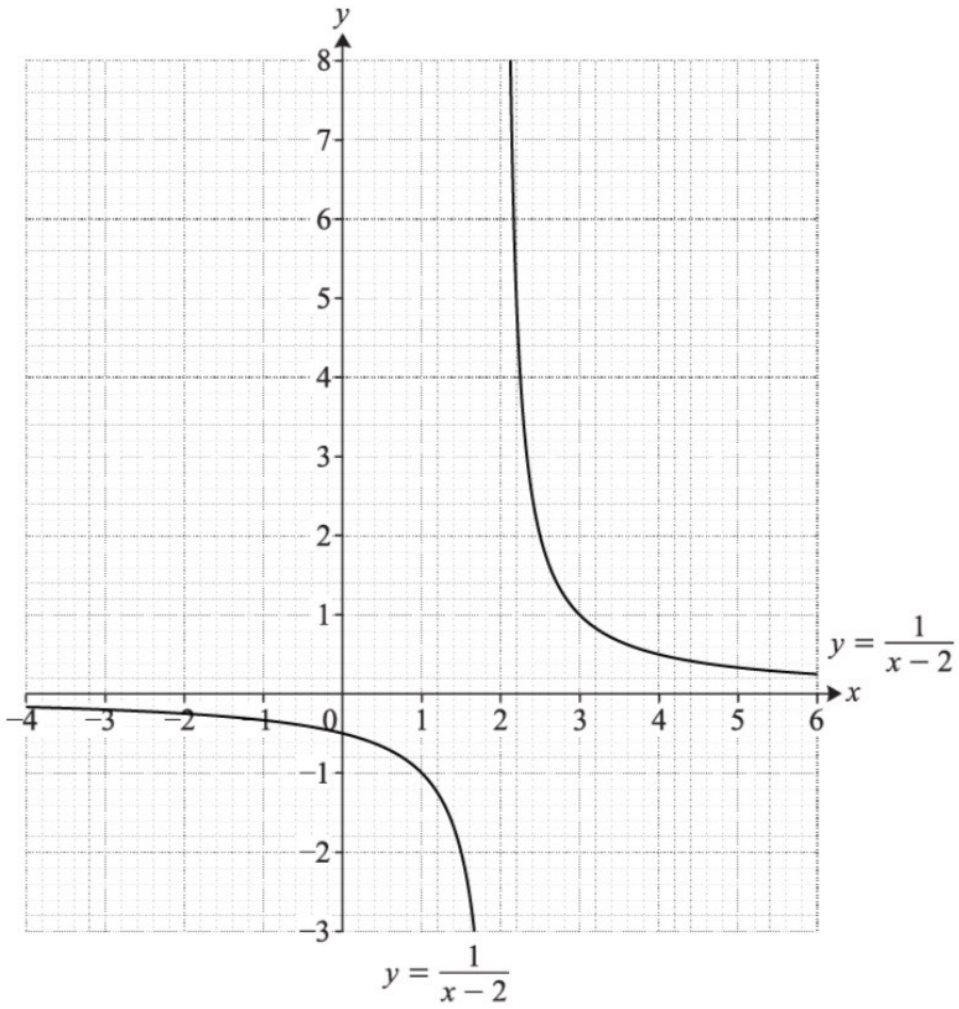


Fig. 12

Fig. 12 shows the graph of $y = \frac{1}{x-2}$.

i. Draw accurately the graph of $y = 2x + 3$ on the copy of Fig. 12 and use it to estimate the coordinates of the points of intersection of $y = \frac{1}{x-2}$ and $y = 2x + 3$.

[3]

ii. Show algebraically that the x-coordinates of the points of intersection of $y = \frac{1}{x-2}$ and $y = 2x + 3$ satisfy the equation $2x^2 - x - 7 = 0$. Hence find the exact values of the x-coordinates of the points of intersection.

[5]

iii. Find the quadratic equation satisfied by the x-coordinates of the points of intersection of $y = \frac{1}{x-2}$ and $y = -x + k$. Hence find the exact values of k for which $y = -x + k$ is a tangent to $y = \frac{1}{x-2}$.

[4]

a Describe fully the transformation that maps the graph of $y = f(x)$ onto the graph of $y = f(x + 1)$. **(2)**

b Sketch the graph of $y = \frac{1}{x+1}$, showing the coordinates of any points of intersection with the coordinate axes and the equations of any asymptotes. **(3)**

c By sketching another suitable curve on your diagram in part **b**, show that the equation

$$x^3 - \frac{1}{x+1} = 2$$

has one positive and one negative real root. **(4)**

a On the same axes sketch the curves given by $y = x^2(x + a)$ and $y = \frac{b}{x}$ where a and b are both positive constants. **(5 marks)**

b Using your sketch, state, giving a reason, the number of real solutions to the equation $x^4 + ax^3 - b = 0$. **(1 mark)**

The graph of $y = f(x)$ where $f(x) = \frac{1}{x}$ is translated so that the asymptotes are at $x = 4$ and $y = 0$. Write down the equation for the transformed function in the form $y = \frac{1}{x + a}$ **(3 marks)**

Given that $f(x) = \frac{1}{x}$, $x \neq 0$,

a Sketch the graph of $y = f(x) - 2$ and state the equations of the asymptotes. **(3 marks)**

b Find the coordinates of the point where the curve $y = f(x) - 2$ cuts a coordinate axis. **(2 marks)**

c Sketch the graph of $y = f(x + 3)$. **(2 marks)**

d State the equations of the asymptotes and the coordinates of the point where the curve cuts a coordinate axis. **(2 marks)**

The curve C_1 has equation $y = -\frac{a}{x^2}$ where a is a positive constant. The curve C_2 has the equation $y = x^2(3x + b)$ where b is a positive constant.

a Sketch C_1 and C_2 on the same set of axes, showing clearly the coordinates of any point where the curves touch or cross the axes. **(4 marks)**

b Using your sketch state, giving reasons, the number of solutions to the equation $x^4(3x + b) + a = 0$. **(2 marks)**

Language of functions

$$f(x) \equiv x^2 - 4x + 5, \quad x \in \mathbb{R}, \quad x \geq 2.$$

a Express $f(x)$ in the form $a(x + b)^2 + c$. **(2)**

b State the range of f. **(1)**

c Find an expression for $f^{-1}(x)$ and state its domain. **(4)**

d Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same diagram and state the relationship between the graphs. **(4)**

A function f is defined by $f(x) = 2e^{3x} - 1$ for all real values of x.

(a) Find the range of f. *(2 marks)*

(b) Show that $f^{-1}(x) = \frac{1}{3}\ln\left(\frac{x+1}{2}\right)$. *(3 marks)*

(c) Find the gradient of the curve $y = f^{-1}(x)$ when $x = 0$. *(4 marks)*

The functions f and g are defined with their respective domains by

$$f(x) = 3 - x^2, \quad \text{for all real values of } x$$

$$g(x) = \frac{2}{x+1}, \quad \text{for real values of } x, \ x \neq -1$$

(a) Find the range of f. *(2 marks)*

(b) The inverse of g is g^{-1}.

(i) Find $g^{-1}(x)$. *(3 marks)*

(ii) State the range of g^{-1}. *(1 mark)*

(c) The composite function gf is denoted by h.

(i) Find $h(x)$, simplifying your answer. *(2 marks)*

(ii) State the greatest possible domain of h. *(1 mark)*

The functions f and g are defined with their respective domains by

$$f(x) = \sqrt{x-2} \quad \text{for } x \geqslant 2$$

$$g(x) = \frac{1}{x} \quad \text{for real values of } x, \ x \neq 0$$

(a) State the range of f. *(2 marks)*

(b) (i) Find $fg(x)$. *(1 mark)*

(ii) Solve the equation $fg(x) = 1$. *(3 marks)*

(c) The inverse of f is f^{-1}. Find $f^{-1}(x)$. *(3 marks)*

Functions f and g are defined by

$f: x \mapsto 4 - x, \quad \{x \in \mathbb{R}\}$

$g: x \mapsto 3x^2, \quad \{x \in \mathbb{R}\}$

a Write down the range of g. **(1 mark)**

b Solve $gf(x) = 48$. **(4 marks)**

c Sketch the graph of $y = |f(x)|$ and hence find the values of x for which $|f(x)| = 2$. **(4 marks)**

The functions f and g are given by

$$f: x \mapsto \frac{x}{x^2-1} - \frac{1}{x+1}, \quad \{x \in \mathbb{R}, x > 1\}$$

$$g: x \mapsto \frac{2}{x}, \quad \{x \in \mathbb{R}, x > 0\}$$

a Show that $f(x) = \dfrac{1}{(x-1)(x+1)}$ **(3 marks)**

b Find the range of $f(x)$. **(1 mark)**

c Solve $gf(x) = 70$. **(4 marks)**

The functions f and g are given by

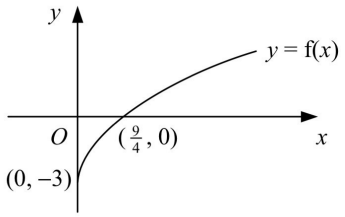
$$f: x \mapsto \frac{x}{x^2-1} - \frac{1}{x+1}, \quad \{x \in \mathbb{R}, x > 1\}$$

$$g: x \mapsto \frac{2}{x}, \quad \{x \in \mathbb{R}, x > 0\}$$

a Show that $f(x) = \dfrac{1}{(x-1)(x+1)}$ **(3 marks)**

b Find the range of $f(x)$. **(1 mark)**

c Solve $gf(x) = 70$. **(4 marks)**



The diagram shows the graph of $y = f(x)$ which meets the x-axis at the point $(\frac{9}{4}, 0)$ and the y-axis at the point $(0, -3)$.

a Sketch on separate diagrams the graphs of

i $y = |f(x)|$,

ii $y = f^{-1}(x)$. **(4)**

Given that $f(x)$ is of the form $f(x) \equiv ax^{\frac{1}{2}} + b, \ x \in \mathbb{R}, \ x \geq 0$,

b find the values of the constants a and b, **(3)**

c find an expression for $f^{-1}(x)$. **(3)**

$$f: x \to 2 + \log_4 x, \ x \in \mathbb{R}, \ x > 0.$$

a Evaluate $ff(1)$. **(3)**

b Solve the equation $f(x) = 0$. **(2)**

c Find the inverse function $f^{-1}(x)$. **(3)**

The function f is defined by

$$f: x \to \frac{x+b}{x+a}, \ x \in \mathbb{R}, \ x \neq 2.$$

a State the value of the constant a.

Given that $f(6) = 4$,

b find the value of the constant b,

c find $f^{-1}(x)$ and state its domain.

The functions f and g are defined by

$$f: x \to x^2 - 3x,\ x \in \mathbb{R},\ x \geq 1\tfrac{1}{2},$$

$$g: x \to 2x + 3,\ x \in \mathbb{R}.$$

a Find, in the form $f^{-1}: x \to \ldots$, the inverse function of f and state its domain.

b On the same set of axes, sketch $y = f(x)$ and $y = f^{-1}(x)$.

Given that $f^{-1}g^{-1}(12) = a(1 + \sqrt{3})$,

c show that $a = 1\frac{1}{2}$.

The functions f and g are defined by

$$f: x \to kx + 2,\ x \in \mathbb{R},$$

$$g: x \to x - 3k,\ x \in \mathbb{R},$$

where k is a constant.

a Find expressions in terms of k for

i $f^{-1}(x)$,

ii $fg(x)$. **(4)**

Given that $fg(7) = 4$,

b find the two possible values of k. **(3)**

The function f is given by

$$f: x \to e^{\frac{1}{2}x} - 3,\ x \in \mathbb{R}.$$

a Find $f^{-1}(x)$ and state its domain. **(4)**

b Sketch the curve $y = f^{-1}(x)$, showing the coordinates of any points of intersection with the coordinate axes. **(3)**

The function g is given by

$$g: x \to \ln(x + 5),\ x \in \mathbb{R},\ x > -5.$$

c Evaluate $fg(4)$. **(2)**

d Solve the equation $f^{-1}(x) = g(x)$. **(4)**

The function f is defined by

$$f: x \to 3 + \ln(x + 2),\ x \in \mathbb{R},\ x \geq k,$$

where k is a constant.

Given that the range of f is $f(x) \geq 3$,

a find the value of k, **(3)**

b find $f^{-1}(x)$, stating its domain clearly. **(4)**

The function g is defined by

$$g: x \to 4 + \ln(x - 1),\ x \in \mathbb{R},\ x > 1.$$

c Find, in terms of e, the value of x such that $f(x) = g(x)$. **(4)**

The functions f and g are given by

$$f(x) \equiv \frac{x}{x+2},\ x \in \mathbb{R},\ x \neq -2,$$

$$g(x) \equiv \frac{3}{x},\ x \in \mathbb{R},\ x \neq 0$$

a Solve the equation $fg(x) = 4$. **(4)**

b Find $f^{-1}(x)$, stating its domain clearly. **(4)**

c Solve the equation $f(x) = f^{-1}(x)$. **(3)**

The function f is defined by

$$\mathrm{f}(x) \equiv x^2 - 2x - 9,\ x \in \mathbb{R},\ x \geq k.$$

a Find the minimum value of the constant k for which $\mathrm{f}^{-1}(x)$ exists. **(3)**

Given that k takes the value found in part **a**,

b solve the equation $\mathrm{f}^{-1}(x) = 4$, **(2)**

c sketch the curve $y = |\mathrm{f}(x)|$, **(3)**

d find the values of x for which $|\mathrm{f}(x)| = 6$. **(5)**

The function f is defined by

$$\mathrm{f} : x \to 2 - \frac{3}{x},\ x \in \mathbb{R},\ x \neq 0.$$

a Find the value of ff(1). **(2)**

b Find $\mathrm{f}^{-1}(x)$ and state its domain. **(4)**

The function g is defined by

$$\mathrm{g} : x \to x^2,\ x \in \mathbb{R}.$$

c Solve the equation $\mathrm{gf}(x) = 1$. **(4)**

The function f is defined by

$$\mathrm{f} : x \to \mathrm{e}^{\frac{1}{2}x} - 2,\ x \in \mathbb{R}.$$

a Evaluate f(ln 9). **(2)**

b State the range of f. **(1)**

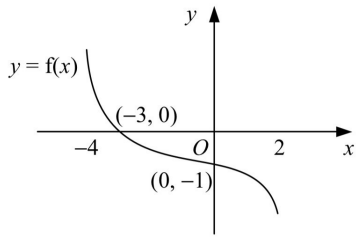
c Find $\mathrm{f}^{-1}(x)$ and state its domain. **(4)**

The function g is defined by

$$\mathrm{g} : x \to x^2 + 4x,\ x \in \mathbb{R}.$$

d Find and simplify an expression for $\mathrm{gf}(x)$. **(3)**

e Solve the equation $\mathrm{gf}(x) + 1 = 0$. **(2)**



The diagram shows the curve $y = \mathrm{f}(x)$. The domain of f is $-4 \leq x \leq 2$ and the curve intersects the coordinate axes at the points $(-3, 0)$ and $(0, -1)$.

a Explain how the graph shows that f is one-one. **(1)**

b Showing the coordinates of any points of intersection with the axes, sketch on separate diagrams the graphs of

i $y = |\mathrm{f}(x)|$,

ii $y = \mathrm{f}^{-1}(x)$. **(5)**

$$\mathrm{f}(x) \equiv \frac{5}{(x+1)(2x-3)} + \frac{1}{x+1},\ x \in \mathbb{R},\ x \geq 2.$$

a Show that $\mathrm{f}(x) = \frac{2}{2x-3}$. **(4)**

b Find the range of f. **(2)**

c Find an expression for $\mathrm{f}^{-1}(x)$. **(3)**

$$\mathrm{g}(x) \equiv \frac{1}{x-2},\ x \in \mathbb{R},\ x \neq 2.$$

d Solve the equation $\mathrm{fg}(x) = \frac{2}{3}$. **(4)**

7 A circle has equation $x^2 + y^2 - 4x - 14 = 0$.

(a) Find:

(i) the coordinates of the centre of the circle; *(3 marks)*

(ii) the radius of the circle in the form $p\sqrt{2}$, where p is an integer. *(3 marks)*

(b) A chord of the circle has length 8. Find the perpendicular distance from the centre of the circle to this chord. *(3 marks)*

(c) A line has equation $y = 2k - x$, where k is a constant.

(i) Show that the x-coordinate of any point of intersection of the line and the circle satisfies the equation

$$x^2 - 2(k+1)x + 2k^2 - 7 = 0 \quad \textit{(3 marks)}$$

(ii) Find the values of k for which the equation

$$x^2 - 2(k+1)x + 2k^2 - 7 = 0$$

has equal roots. *(4 marks)*

(iii) Describe the geometrical relationship between the line and the circle when k takes either of the values found in part (c)(ii). *(1 mark)*

A circle with centre C has equation $x^2 + y^2 + 2x - 12y + 12 = 0$.

(a) By completing the square, express this equation in the form

$$(x-a)^2 + (y-b)^2 = r^2 \quad \textit{(3 marks)}$$

(b) Write down:

(i) the coordinates of C; *(1 mark)*

(ii) the radius of the circle. *(1 mark)*

(c) Show that the circle does **not** intersect the x-axis. *(2 marks)*

(d) The line with equation $x + y = 4$ intersects the circle at the points P and Q.

(i) Show that the x-coordinates of P and Q satisfy the equation

$$x^2 + 3x - 10 = 0 \quad \textit{(3 marks)}$$

(ii) Given that P has coordinates $(2, 2)$, find the coordinates of Q. *(2 marks)*

(iii) Hence find the coordinates of the midpoint of PQ. *(2 marks)*

The circle C has equation $x^2 + y^2 - 8x - 16y + 72 = 0$.

a Find the coordinates of the centre and the radius of C. **(3)**

b Find the distance of the centre of C from the origin in the form $k\sqrt{5}$. **(2)**

The point A lies on C and the tangent to C at A passes through the origin O.

c Show that $OA = 6\sqrt{2}$. **(3)**

The circle C has equation $x^2 + y^2 - 4x - 6 = 0$ and the line l has equation $y = 3x - 6$.

a Show that l passes through the centre of C. **(3)**

b Find an equation for each tangent to C that is parallel to l. **(6)**

A circle with centre C has equation $(x+3)^2+(y-2)^2=25$.

(a) Write down:

(i) the coordinates of C; *(2 marks)*

(ii) the radius of the circle. *(1 mark)*

(b) (i) Verify that the point $N(0, -2)$ lies on the circle. *(1 mark)*

(ii) Sketch the circle. *(2 marks)*

(iii) Find an equation of the normal to the circle at the point N. *(3 marks)*

(c) The point P has coordinates (2, 6).

(i) Find the distance PC, leaving your answer in surd form. *(2 marks)*

(ii) Find the length of a tangent drawn from P to the circle. *(3 marks)*

A circle has the equation $x^2+y^2-6x+2y-15=0$.

a Find the coordinates of the centre of the circle. **(2)**

b Find the radius of the circle. **(1)**

c Show that the tangent to the circle at the point (7, 2) has equation

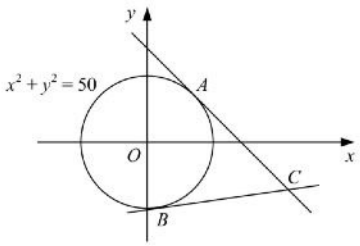
$4x+3y-34=0$. **(4)**

A circle has the equation $x^2+y^2+6x-8y+21=0$.

a Find the coordinates of the centre and the radius of the circle. **(3)**

The point P lies on the circle.

b Find the greatest distance of P from the origin. **(2)**



The diagram shows the circle with equation $x^2+y^2=50$ and the tangents to the circle at the points A (5, 5) and B (1, −7).

a Find an equation of the tangent to the circle at A. **(3)**

b Show that the tangent to the circle at B has the equation

$x-7y-50=0$. **(3)**

c Find the coordinates of the point C where the tangents to the circle at A and B intersect. **(2)**

The circle C has the equation $x^2+y^2+2x-14y+30=0$.

a Find the coordinates of the centre of C. **(2)**

b Find the radius of C, giving your answer in the form $k\sqrt{5}$. **(2)**

c Show that the line $y=2x-1$ is a tangent to C and find the coordinates of the point of contact. **(4)**

The circle C has equation $x^2+y^2-6x-12y+28=0$.

a Find the coordinates of the centre of C. **(2)**

The line $y=x-2$ intersects C at the points A and B.

b Find the length AB in the form $k\sqrt{2}$. **(6)**

The circle C has centre $(8, -1)$ and passes through the point $(4, 1)$.

a Find an equation for C. **(3)**

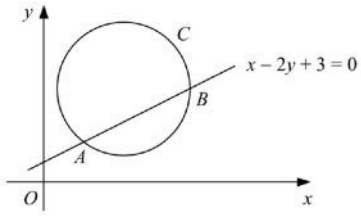
b Show that the line with equation $x + 2y + 4 = 0$ is a tangent to C. **(3)**

A circle has the equation $x^2 + y^2 - 2x - 7y - 16 = 0$.

a Find the coordinates of the centre of the circle. **(2)**

b Show that the radius of the circle is $k\sqrt{13}$, where k is an exact fraction to be found. **(2)**

c Find an equation of the tangent to the circle at the point $(4, 8)$, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers. **(4)**



The line with equation $x - 2y + 3 = 0$ intersects the circle C at the points A and B as shown in the diagram above. Given that the centre of C has coordinates $(6, 7)$,

a find the coordinates of the mid-point of the chord AB. **(6)**

Given also that the x-coordinate of the point A is 3,

b find the coordinates of the point B, **(3)**

c find an equation for C. **(2)**

Binomial expansion; area under a curve

Find the first four terms of the binomial expansion of $(1-2x)^{\frac{1}{2}}$. [4]

State the set of values of x for which the expansion is valid.

i. Find the first three terms of the binomial expansion of $\frac{1}{\sqrt[3]{1-2x}}$. State the set of values of x for which the expansion is valid.

[5]

ii. Hence find a and b such that $\frac{1-3x}{\sqrt[3]{1-2x}} = 1 + ax + bx^2 + \ldots$.

[3]

Find the first four terms of the binomial expansion of $\sqrt[3]{1-2x}$. State the set of values of x for which the expansion is valid.

[6]

Find the first three terms in the binomial expansion of $(4+x)^{\frac{3}{2}}$. State the set of values of x for which the expansion is valid.

[5]

(a) Use the binomial expansion to show that $(1-2x)^{-\frac{1}{2}} \approx 1+x+\frac{3}{2}x^2$ for sufficiently small values of x. [2

(b) For what values of x is the expansion valid? [1

(c) Find the expansion of $\sqrt{\frac{1+2x}{1-2x}}$ in ascending powers of x as far as the term in x^2. [3

(d) Use $x = \frac{1}{20}$ in your answer to part (c) to find an approximate value for $\sqrt{11}$. [2

(i) Express $\frac{5-x}{(2-x)(1+x)}$ in partial fractions. [3]

(ii) Hence or otherwise find the first 3 terms of the binomial expansion of $\frac{5-x}{(2-x)(1+x)}$ in ascending powers of x. [5]

In this question you must show detailed reasoning.

Given that

$$(1 + ax)^n = 1 + 6x - 6x^2 + \ldots,$$

where a and n are constants, find the values of a and n. [6]

(a) Find the first four terms in the expansion of $\left(1+\frac{x}{2}\right)^{-2}$. [3]

(b) State the range of values of x for which this expansion is valid. [1]

(a) Find the first 4 terms, in ascending powers of x, of the binomial expansion of $(1 + 3x)^{-1}$ [3]

(b) State the range of values of x for which this expansion is valid. [1]

i. Express $\frac{x}{(1+x)(1-2x)}$ in partial fractions.

[3]

ii. Hence use binomial expansions to show that $\frac{x}{(1+x)(1-2x)} = ax + bx^2 + \ldots,$ where a and b are constants to be determined.

State the set of values of x for which the expansion is valid.

[5]

(a) Find and simplify the first three terms in the expansion of $\left(3-\frac{x}{2}\right)^6$ in ascending powers of x. [3]

(b) Explain how the result in part (a) can be used to give an approximation to the value of $(2.95)^6$. [1]

i. Find and simplify the first three terms in the expansion of $(2 + 5x)^6$ in ascending powers of x.

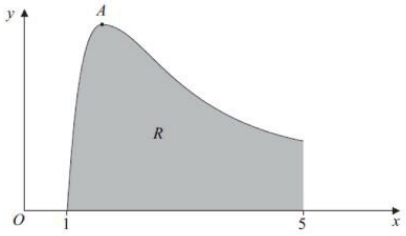
[4]

ii. In the expansion of $(3 + cx)^2(2 + 5x)^6$, the coefficient of x is 4416. Find the value of c.

[3]

(b) Using integration by parts, find $\int x^{-2} \ln x \, dx$. *(4 marks)*

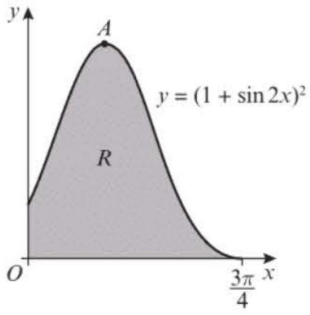
(c) The sketch shows the graph of $y = x^{-2} \ln x$.



(ii) The region R is bounded by the curve, the x-axis and the line $x = 5$. Using your answer to part (b), show that the area of R is

$\frac{1}{5}(4 - \ln 5)$ *(3 marks)*

The diagram shows the graph of $y = (1 + \sin 2x)^2$, $0 \leqslant x \leqslant \frac{3\pi}{4}$

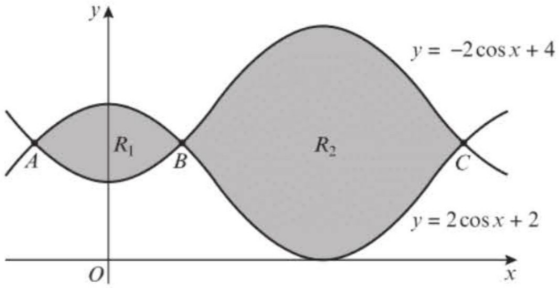


a Show that $(1 + \sin 2x)^2 \equiv \frac{1}{2}(3 + 4\sin 2x - \cos 4x)$. **(4 marks)**

b Hence find the area of the shaded region R. **(4 marks)**

c Find the coordinates of A, the turning point on the graph. **(3 marks)**

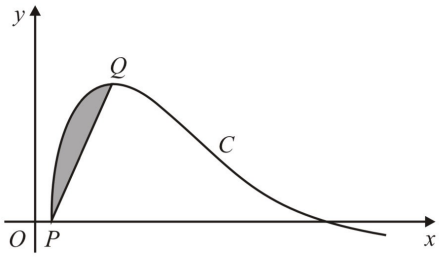
The diagram shows a sketch of part of the curves with equations $y = 2\cos x + 2$ and $y = -2\cos x + 4$.



a Find the coordinates of the points A, B and C. **(2 marks)**

b Find the area of region R_1 in the form $a\sqrt{3} + \frac{b\pi}{c}$, where a, b and c are integers to be found. **(4 marks)**

c Show that the ratio of $R_2 : R_1$ can be expressed as $(3\sqrt{3} + 2\pi) : (3\sqrt{3} - \pi)$. **(5 marks)**



The figure above shows a sketch of part of the curve C with equation

$$y = \sin(\ln x), \qquad x \geq 1.$$

The point Q, on C, is a maximum.

(a) Show that the point $P(1, 0)$ lies on C.

(1)

(b) Find the coordinates of the point Q.

(5)

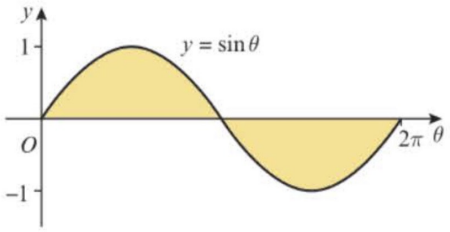
(c) Find the area of the shaded region between C and the line PQ.

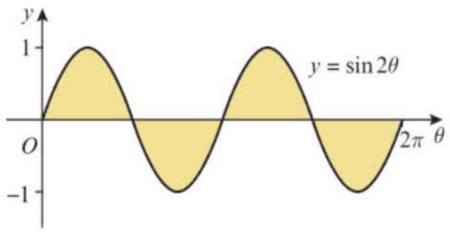
(9)

(Total 15 marks)

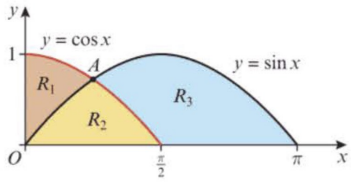
The diagrams show the curves $y = \sin\theta$, $0 \leqslant \theta \leqslant 2\pi$ and $y = \sin 2\theta$, $0 \leqslant \theta \leqslant 2\pi$.

By choosing suitable limits, show that the total shaded area in the first diagram is equal to the total shaded area in the second diagram, and state the exact value of this shaded area.





The diagram shows parts of the graphs of $y = \sin x$ and $y = \cos x$.



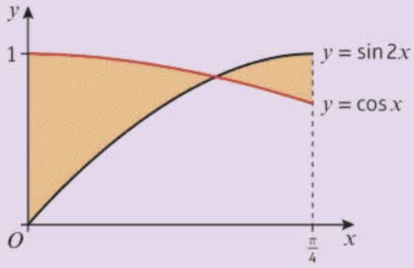
a Find the coordinates of point A.

b Find the areas of:

i R_1 **ii** R_2 **iii** R_3

c Show that the ratio of areas $R_1 : R_2$ can be written as $\sqrt{2} : 2$.

The diagram shows the curves $y = \sin 2x$ and $y = \cos x$, $0 \leqslant x \leqslant \frac{\pi}{4}$



Find the exact value of the total shaded area on the diagram.

Sequences from formulas

A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = k,$$

$$a_{n+1} = 2a_n - 7, \quad n \geq 1,$$

where k is a constant.

(a) Write down an expression for a_2 in terms of k.

(1)

(b) Show that $a_3 = 4k - 21$.

(2)

Given that $\sum_{r=1}^{4} a_r = 43$,

(c) find the value of k.

(4)

(Total 7 marks)

A sequence of positive numbers is defined by

$$a_{n+1} = \sqrt{(a_n^2 + 3)}, \qquad n \geq 1,$$

$$a_1 = 2$$

(a) Find a_2 and a_3, leaving your answers in surd form.

(2)

(b) Show that $a_5 = 4$

(2)
(Total 4 marks)

A sequence is defined by the recurrence relation

$$u_{n+1} = \sqrt{\left(\frac{u_n}{2} + \frac{a}{u_n}\right)}, \quad n = 1, 2, 3, \ldots,$$

where a is a constant.

(a) Given that $a = 20$ and $u_1 = 3$, find the values of u_2, u_3 and u_4, giving your answers to 2 decimal places.

(3)

(b) Given instead that $u_1 = u_2 = 3$,

(i) calculate the value of a,

(3)

(ii) write down the value of $u5$.

(1)
(Total 7 marks)

A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = k, \quad a_{n+1} = 4a_n - 7,$$

where k is a constant.

(a) Write down an expression for a_2 in terms of k.

(1)

(b) Find a_3 in terms of k, simplifying your answer.

(2)

Given that $a_3 = 13$,

(c) find the value of k.

(2)
(Total 5 marks)

A sequence $x_1, x_2, x_3, \ldots$ is defined by

$$x_1 = 1,$$

$$x_{n+1} = ax_n - 3, n > 1,$$

where a is a constant.

(a) Find an expression for x_2 in terms of a.

(1)

(b) Show that $x_3 = a_2 - 3a - 3$.

(2)

Given that $x_3 = 7$,

(c) find the possible values of a.

(3)
(Total 6 marks)

A sequence is given by:

$$x_1 = 1,$$

$$x_{n+1} = x_n(p + x_n),$$

where p is a constant ($p \neq 0$).

(a) Find x_2 in terms of p.

(1)

(b) Show that $x_3 = 1 + 3p + 2p^2$.

(2)

Given that $x_3 = 1$,

(c) find the value of p,

(3)

(d) write down the value of x_{2008}.

(2)
(Total 8 marks)

A sequence $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = 3$$

$$a_{n+1} = 3a_n - 5, \ n \geq 1.$$

(a) Find the value of a_2 and the value of a_3.

(2)

(b) Calculate the value of $\sum_{r=1}^{5} a_r$

(3)
(Total 5 marks)

A sequence $a_1, a_2, a_3 \ldots$, is defined by

$$a_1 = k,$$

$$a_{n+1} = 3a_n + 5, \qquad n \geq 1,$$

where k is a positive integer.

(a) Write down an expression for a_2 in terms of k.

(1)

(b) Show that $a_3 = 9k + 20$.

(2)

(c) (i) Find $\sum_{r=1}^{4} a_r$ in terms of k.

(ii) Show that $\sum_{r=1}^{4} a_r$ is divisible by 10.

(4)
(Total 7 marks)

The sequence $u_1, u_2, u_3, \ldots$, is defined by the recurrence relation

$$u_{n+1} = (-1)^n u_n + d, \quad u_1 = 2, \quad \text{where } d \text{ is a constant.}$$

(a) Show that $u_5 = 2$.

(4)

(b) Deduce an expression for u_{10}, in terms of d.

(1)

Given that $u_3 = 3u_2$,

(c) find the value of d.

(2)
(Total 7 marks)

Trigonometric equations

(a) Express $5\cos x - 3\sin x$ in the form $R\cos(x + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$.

(4)

(b) Hence, or otherwise, solve the equation

$$5\cos x - 3\sin x = 4$$

for $0 \leq x < 2\pi$, giving your answers to 2 decimal places.

(5)
(Total 9 marks)

Solve

$$\text{cosec}^2 2x - \cot 2x = 1$$

for $0 \leq x \leq 180°$.

(Total 7 marks)

(a) Use the identity $\cos^2\theta + \sin^2\theta = 1$ to prove that $\tan^2\theta = \sec^2\theta - 1$.

(2)

(b) Solve, for $0 \leq \theta < 360°$, the equation

$$2\tan^2\theta + 4\sec\theta + \sec^2\theta = 2$$

(6)
(Total 8 marks)

(a) Use the identity $\cos(A + B) = \cos A\cos B - \sin A\sin B$, to show that

$$\cos 2A = 1 - 2\sin^2 A$$

(2)

The curves C_1 and have equations

$$C_1: y = 3\sin 2x$$

$$y = 4\sin^2 x - 2\cos 2x$$

(b) Show that the x-coordinates of the points where C_1 and intersect satisfy the equation

$$4\cos 2x + 3\sin 2x = 2$$

(3)

(c) Express $4\cos 2x + 3\sin 2x$ in the form $R\cos(2x - \alpha)$, where $R > 0$ and $0 < \alpha < 90°$, giving the value of α to 2 decimal places.

(3)

(d) Hence find, for $0 \leq x < 180°$, all the solutions of

$$4\cos 2x + 3\sin 2x = 2$$

giving your answers to 1 decimal place.

(4)
(Total 12 marks)

(a) Write down sin $2x$ in terms of sin x and cos x.

(1)

(b) Find, for $0 < x < \pi$, all the solutions of the equation

$$\text{cosec } x - 8 \cos x = 0$$

giving your answers to 2 decimal places.

(5)
(Total 6 marks)

(a) (i) By writing $3\theta = (2\theta + \theta)$, show that

$$\sin 3\theta = 3 \sin\theta - 4 \sin^3\theta.$$

(4)

(ii) Hence, or otherwise, for $0 < \theta < \frac{\pi}{3}$, solve

$$8 \sin^3\theta - 6 \sin \theta + 1 = 0.$$

Give your answers in terms of π.

(5)

(b) Using $\sin(\theta - \alpha) = \sin\theta\cos\alpha - \cos\theta\sin\alpha$, or otherwise, show that

$$\sin 15° = \frac{1}{4}(\sqrt{6} - \sqrt{2}).$$

(4)
(Total 13 marks)

(a) Given that $\sin^2\theta + \cos^2\theta \equiv 1$, show that $1 + \cot^2\theta \equiv \text{cosec}^2\theta$.

(2)

(b) Solve, for $0 \leq \theta < 180°$, the equation

$$2\cot^2\theta - 9\text{cosec}\theta = 3,$$

giving your answers to 1 decimal place.

(6)
(Total 8 marks)

(a) Using $\sin^2\theta + \cos^2\theta \equiv 1$, show that $\text{cosec}^2\theta - \cot^2\theta \equiv 1$.

(2)

(b) Hence, or otherwise, prove that

$$\text{cosec}^4\theta - \cot^4\theta \equiv \text{cosec}^2\theta + \cot^2\theta.$$

(2)

(c) Solve, for $90° < \theta < 180°$,

$$\text{cosec}^4\theta - \cot^4\theta = 2 - \cot\theta.$$

(6)
(Total 10 marks)

(a) Show that

(i) $\dfrac{\cos 2x}{\cos x + \sin x} \equiv \cos x - \sin x, \qquad x \neq (n - \frac{1}{4})\pi, \ n \in \mathbb{Z}$

(2)

(ii) $\frac{1}{2}(\cos 2x - \sin 2x) \equiv \cos^2 x - \cos x \sin x - \frac{1}{2}$

(3)

(b) Hence, or otherwise, show that the equation

$$\cos\theta\left(\frac{\cos 2\theta}{\cos\theta + \sin\theta}\right) = \frac{1}{2}$$

can be written as

$$\sin 2\theta = \cos 2\theta.$$

(3)

(c) Solve, for $0 \leq \theta \leq 2\pi$,

$$\sin 2\theta = \cos 2\theta,$$

giving your answers in terms of π.

(4)
(Total 12 marks)

$f(x) = 12 \cos x - 4 \sin x$.

Given that $f(x) = R \cos(x + \alpha)$, where $R \geq 0$ and $0 \leq \alpha \leq 90°$,

(a) find the value of R and the value of α.

(4)

(b) Hence solve the equation

$$12 \cos x - 4 \sin x = 7$$

for $0 \leq x \leq 360°$, giving your answers to one decimal place.

(5)

(c) (i) Write down the minimum value of $12 \cos x - 4 \sin x$.

(1)

(ii) Find, to 2 decimal places, the smallest positive value of x for which this minimum value occurs.

(2)
(Total 12 marks)

(a) Given that $2\sin(\theta + 30)° = \cos(\theta + 60)°$, find the exact value of $\tan\theta°$.

(5)

(b) (i) Using the identity $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$, prove that

$$\cos 2A \equiv 1 - 2\sin^2 A.$$

(2)

(ii) Hence solve, for $0 \leq x < 2\pi$,

$$\cos 2x = \sin x,$$

giving your answers in terms of π.

(5)

(iii) Show that $\sin 2y \tan y + \cos 2y \equiv 1$, for $0 \leq y < \frac{1}{2}\pi$.

(3)

(Total 15 marks)

(a) Given that $\sin^2\theta + \cos^2\theta \equiv 1$, show that $1 + \tan^2\theta \equiv \sec^2\theta$.

(2)

(b) Solve, for $0 \leq \theta < 360°$, the equation

$$2\tan^2\theta + \sec\theta = 1,$$

giving your answers to 1 decimal place.

(6)

(Total 8 marks)

(a) Using the identity $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$, prove that

$$\cos 2A \equiv 1 - 2\sin^2 A.$$

(2)

(b) Show that

$$2\sin 2\theta - 3\cos 2\theta - 3\sin\theta + 3 \equiv \sin\theta(4\cos\theta + 6\sin\theta - 3).$$

(4)

(c) Express $4\cos\theta + 6\sin\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$.

(4)

(d) Hence, for $0 \leq \theta < \pi$, solve

$$2\sin 2\theta = 3(\cos 2\theta + \sin\theta - 1),$$

giving your answers in radians to 3 significant figures, where appropriate.

(5)

(Total 15 marks)

(a) Sketch, on the same axes, in the interval $0 \leq x \leq 180$, the graphs of

$y = \tan x°$ and $y = 2 \cos x°$,

showing clearly the coordinates of the points at which the graphs meet the axes.

(4)

(b) Show that $\tan x° = 2 \cos x°$ can be written as

$2 \sin^2 x° + \sin x° - 2 = 0$.

(3)

(c) Hence find the values of x, in the interval $0 \leq x \leq 180$, for which $\tan x° = 2 \cos x°$.

(4)

(Total 11 marks)

(i) (a) Express $(12 \cos \theta - 5 \sin \theta)$ in the form $R \cos (\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90°$.

(4)

(b) Hence solve the equation

$$12 \cos \theta - 5 \sin \theta = 4,$$

for $0 < \theta < 90°$, giving your answer to 1 decimal place.

(3)

(ii) Solve

$$8 \cot \theta - 3 \tan \theta = 2,$$

for $0 < \theta < 90°$, giving your answer to 1 decimal place.

(5)

(Total 12 marks)

(a) Use the double angle formulae and the identity

$$\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$$

to obtain an expression for $\cos 3x$ in terms of powers of $\cos x$ only.

(4)

(b) (i) Prove that

$$\frac{\cos x}{1 + \sin x} + \frac{1 + \sin x}{\cos x} \equiv 2 \sec x, \quad x \neq (2n + 1)\frac{\pi}{2}.$$

(4)

(ii) Hence find, for $0 < x < 2\pi$, all the solutions of

$$\frac{\cos x}{1 + \sin x} + \frac{1 + \sin x}{\cos x} = 4.$$

(3)

(Total 11 marks)

(a) Using $\sin^2\theta + \cos^2\theta \equiv 1$, show that $\text{cosec}^2\theta - \cot^2\theta \equiv 1$.

(2)

(b) Hence, or otherwise, prove that

$$\text{cosec}^4\theta - \cot^4\theta \equiv \text{cosec}^2\theta + \cot^2\theta.$$

(2)

(c) Solve, for $90° < \theta < 180°$,

$$\text{cosec}^4\theta - \cot^4\theta = 2 - \cot\theta.$$

(6)
(Total 10 marks)

(a) Show that

(i) $\dfrac{\cos 2x}{\cos x + \sin x} \equiv \cos x - \sin x, \qquad x \neq (n - \frac{1}{4})\pi, \ n \in \mathbb{Z}$

(2)

(ii) $\frac{1}{2}(\cos 2x - \sin 2x) \equiv \cos^2 x - \cos x \sin x - \frac{1}{2}$

(3)

(b) Hence, or otherwise, show that the equation

$$\cos\theta\left(\frac{\cos 2\theta}{\cos\theta + \sin\theta}\right) = \frac{1}{2}$$

can be written as

$$\sin 2\theta = \cos 2\theta.$$

(3)

(c) Solve, for $0 \leq \theta \leq 2\pi$,

$$\sin 2\theta = \cos 2\theta,$$

giving your answers in terms of π.

(4)
(Total 12 marks)

(a) Using the identity $\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$, prove that

$$\cos 2A \equiv 1 - 2\sin^2 A.$$

(2)

(b) Show that

$$2\sin 2\theta - 3\cos 2\theta - 3\sin\theta + 3 \equiv \sin\theta(4\cos\theta + 6\sin\theta - 3).$$

(4)

(c) Express $4\cos\theta + 6\sin\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$.

(4)

(d) Hence, for $0 \leq \theta < \pi$, solve

$$2\sin 2\theta = 3(\cos 2\theta + \sin\theta - 1),$$

giving your answers in radians to 3 significant figures, where appropriate.

(5)
(Total 15 marks)

(a) Prove that

$$\frac{1-\tan^2\theta}{1+\tan^2\theta} \equiv \cos 2\theta.$$

(4)

(b) Hence, or otherwise, prove

$$\tan^2 \frac{\pi}{8} = 3 - 2\sqrt{2}.$$

(5)
(Total 9 marks)

Implicit differentiation; stationary points

A curve is defined by the equation

$$y^2 - xy + 3x^2 - 5 = 0$$

(a) Find the y-coordinates of the two points on the curve where $x = 1$. *(3 marks)*

(b) (i) Show that $\frac{dy}{dx} = \frac{y - 6x}{2y - x}$. *(6 marks)*

(ii) Find the gradient of the curve at each of the points where $x = 1$. *(2 marks)*

(iii) Show that, at the two stationary points on the curve, $33x^2 - 5 = 0$. *(3 marks)*

The point $P(1, a)$, where $a > 0$, lies on the curve $y + 4x = 5x^2y^2$.

(a) Show that $a = 1$. *(2 marks)*

(b) Find the gradient of the curve at P. *(7 marks)*

(c) Find an equation of the tangent to the curve at P. *(1 mark)*

A curve has the equation

$$3x^2 + xy - y^2 + 9 = 0.$$

Find an expression for $\frac{dy}{dx}$ in terms of x and y. **(5)**

A curve has the equation

$$x^2 - 4xy + y^2 = 24.$$

a Show that $\frac{dy}{dx} = \frac{x - 2y}{2x - y}$. **(4)**

b Find an equation for the tangent to the curve at the point P (2, 10). **(3)**

The tangent to the curve at Q is parallel to the tangent at P.

c Find the coordinates of Q. **(4)**

A curve has the equation

$$2x + x^2y - y^2 = 0.$$

Find the coordinates of the point on the curve where the tangent is parallel to the x-axis. **(8)**

a Express $\frac{8x-18}{(3x-8)(x-2)}$ in partial fractions. **(3 marks)**

b Given that $x \geqslant 3$, find the general solution to the differential equation

$(x-2)(3x-8)\frac{dy}{dx} = (8x-18)y$ **(5 marks)**

c Hence find the particular solution to this differential equation that satisfies $y = 8$ at $x = 3$, giving your answer in the form $y = f(x)$. **(4 marks)**

Find the particular solution to the differential equation $\frac{dy}{dx} = xe^{-y}$, with boundary condition $y = \ln 2$ at $x = 4$. Give your answer in the form $y = f(x)$. **(8 marks)**

Find the general solution to the differential equation $e^{x+y}\frac{dy}{dx} = 2x + xe^{y}$, giving your answer in the form $\ln|g(y)| = f(x)$. **(6 marks)**

Find the particular solution to the differential equation $(1-x^2)\frac{dy}{dx} = xy + y$, with boundary condition $y = 6$ at $x = 0.5$. Give your answer in the form $y = f(x)$. **(8 marks)**

a Express $\frac{1}{(1+x)(1-x)}$ in partial fractions.

In an industrial process, the mass of a chemical, m kg, produced after t hours is modelled by the differential equation

$$\frac{dm}{dt} = ke^{-t}(1+m)(1-m),$$

where k is a positive constant.

Given that when $t = 0$, $m = 0$ and that the initial rate at which the chemical is produced is 0.5 kg per hour,

b find the value of k,

c show that, for $0 \leq m < 1$, $\ln\left(\frac{1+m}{1-m}\right) = 1 - e^{-t}$.

d find the time taken to produce 0.1 kg of the chemical,

e show that however long the process is allowed to run, the maximum amount of the chemical that will be produced is about 462 g.

At time $t = 0$, a piece of radioactive material has mass 24 g. Its mass after t days is m grams and is decreasing at a rate proportional to m.

a By forming and solving a suitable differential equation, show that

$$m = 24e^{-kt},$$

where k is a positive constant.

After 20 days, the mass of the material is found to be 22.6 g.

b Find the value of k.

c Find the rate at which the mass is decreasing after 20 days.

d Find how long it takes for the mass of the material to be halved.

a Express $\frac{x+4}{(1+x)(2-x)}$ in partial fractions.

b Given that $y = 2$ when $x = 3$, solve the differential equation

$$\frac{dy}{dx} = \frac{y(x+4)}{(1+x)(2-x)}.$$

Given that $y = 0$ when $x = 0$, solve the differential equation

$$\frac{dy}{dx} = e^{x+y}\cos x.$$

A quantity has the value N at time t hours and is increasing at a rate proportional to N.

a Write down a differential equation relating N and t.

b By solving your differential equation, show that

$$N = Ae^{kt},$$

where A and k are constants and k is positive.

Given that when $t = 0$, $N = 40$ and that when $t = 5$, $N = 60$,

c find the values of A and k,

d find the value of N when $t = 12$.

The temperature of a freezer is –20°C. A container of a liquid is placed in the freezer. The rate at which the temperature, θ°C, of a liquid decreases is proportional to the difference in temperature between the liquid and its surroundings. The situation is modelled by the differential equation

$$\frac{d\theta}{dt} = -k(\theta + 20),$$

where time t is in minutes and k is a positive constant.

i. Express θ in terms of t, k and an arbitrary constant.

[3]

Initially the temperature of the liquid in the container is 40°C and, at this instant, the liquid is cooling at a rate of 3°C per minute. The liquid freezes at 0°C.

ii. Find the value of k and find also the time it takes (to the nearest minute) for the liquid to freeze.

[5]

The procedure is repeated on another occasion with a different liquid. The initial temperature of this liquid is 90°C. After 19 minutes its temperature is 0°C.

iii. Without any further calculation, explain what you can deduce about the value of k in this case.

[1]

2. At time t seconds, the radius of a spherical balloon is r cm. The balloon is being inflated so that the rate of increase of its radius is inversely proportional to the square root of its radius. When $t = 5$, $r = 9$ and, at this instant, the radius is increasing at 1.08 cm s^{-1}.

i. Write down a differential equation to model this situation, and solve it to express r in terms of t.

[7]

ii. How much air is in the balloon initially?

[2]

[The volume of a sphere is $V = \frac{4}{3}\pi r^3$.]

A container in the shape of an inverted cone of radius 3 metres and vertical height 4.5 metres is initially filled with liquid fertiliser. This fertiliser is released through a hole in the bottom of the container at a rate of 0.01 m^3 per second. At time t seconds the fertiliser remaining in the container forms an inverted cone of height h metres.

[The volume of a cone is $V = \frac{1}{3}\pi r^2 h$.]

i. Show that $h^2 \frac{dh}{dt} = -\frac{9}{400\pi}$.

[5]

ii. Express h in terms of t.

[4]

iii. Find the time it takes to empty the container, giving your answer to the nearest minute.

[2]

In the year 2000 the population density, P, of a village was 100 people per km^2, and was increasing at the rate of 1 person per km^2 per year. The rate of increase of the population density is thought to be inversely proportional to the size of the population density. The time in years after the year 2000 is denoted by t.

i. Write down a differential equation to model this situation, and solve it to express P in terms of t.

[6]

ii. In 2008 the population density of the village was 108 people per km^2 and in 2013 it was 128 people per km^2. Determine how well the model fits these figures.

[2]

Comprehension